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Quantum Mechanics as a Stochastic Process with a $U(1)$ Degree of Freedom[1]

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A stochastic theory equivalent to the nonrelativistic quantum mechanics is constructed. A geometric manifestation of $U(1)$ local gauge invariance is proposed. The stochastic theory is not of the type of Nelson's stochastic mechanics.

1. INTRODUCTION

From the very beginning of the development of the quantum theory there were attempts to interpret the quantum theory as a stochastic process (Furth, 1933; Fenyes, 1952; Kershaw, 1964; Nelson, 1966, 1967; Jammer, 1974). The stochastic quantum theory of Nelson's type (Furth, 1933; Fenyes, 1952; Kershaw, 1964; Nelson, 1966, 1967; Ghirardi et al., 1978) is able to reproduce all the predictions of nonrelativistic quantum mechanics, but it has the following feature: the diffusion drift of the random motion of a particle in the medium is not preassigned, but depends on the wave function (which characterizes an ensemble as a whole). Therefore, the procedure of preparation of the quantum ensemble influences strongly the properties of the underlying medium (see the very comprehensive discussion on this subject in Ghirardi et al., 1978).

In the theory presented here (to be referred to as SPM for stochastic mechanics with phase) there is no influence of the procedure of preparation of the quantum state on the underlying medium. So the stochastic motion of some particle from the ensemble is independent of the other members of the ensemble (as in statistical physics). In this sense the interpretation of

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quantum mechanics given in this paper is a statistical interpretation (see definitions and references in Jammer, 1974, Chap. 10).

There is another difference between Nelson's stochastic mechanics and SPM. In Nelson's theory the state of a particle is described classically by its position in space only. The amount of information contained in the probability density $\rho(t,\mathbf{x})$ and its time derivative $(\partial/\partial t)\,\rho(t,\mathbf{x})$ at some moment t is equal to the amount of information contained in the wave function $\psi(t,\mathbf{x})$. In SPM a particle is described classically by its position $\mathbf{x}$, velocity $\mathbf{v}$, and phase ϕ, an extra variable connected to $U(1)$ gauge degree of freedom. The probability density $\rho(t,\mathbf{x},\mathbf{v},\phi)$ contains more information than $\psi(t,\mathbf{x})$. In this sense we have hidden parameters of the first type in the terminology of Belinfante (1975), but as an advantage we have a natural implementation of the gauge degree of freedom and a geometrical manifestation of gauge invariance.

2. CONSTRUCTION OF STOCHASTIC PROCESS DESCRIBING FREE PARTICLE

We assume that the state of a particle is described by the following parameters.

1. a point in the phase space $(\mathbf{x},\mathbf{v})$,
2. a phase ϕ.

The parameters ϕ will be connected to the $U(1)$ gauge degree of freedom. In order to construct a stochastic process describing a particle moving in a medium we have to define (Gihman and Skorohod, 1974; Dynkin, 1965) the initial probability density $\rho(t_i,\mathbf{x}_i,\mathbf{v}_i,\phi_i)$ and the transition probability kernel $p(t_i,\mathbf{x}_i,\mathbf{v}_i,\phi_i|t_f,\mathbf{x}_f,\mathbf{v}_f,\phi_f)$. We define $p(t_i,x_i,v_i,\phi_i|t_f,x_f,v_f,\phi_f)$ through the Feynman path integral (Feynman and Hibbs, 1965) by the following assumptions:

Assumption 1. All Feynman paths have an equal probability, or in mathematical form

$$p\left(t_i,\mathbf{x}_i,\mathbf{v}_i,\phi_i|t_f,\mathbf{x}_f,\mathbf{v}_f,\phi_f\right)$$

$$= \int_{\mathbf{x}_i(t_i)}^{\mathbf{x}_f(t_f)} \mathscr{D}(\mathbf{x}(t))\,\delta\left(\mathbf{v}_f-\mathbf{v}_i-(\dot{\mathbf{x}}_f-\dot{\mathbf{x}}_i)\right)\delta\left(\phi_f-\phi_i-\frac{1}{\hbar}S\right) \tag{1}$$

where the change of the phase $(1/\hbar)S$ is defined by the following assumption:

Assumption 2. If a particle moves along the path $x(t)$, the change of phase is[2]

$$\phi_f - \phi_i = \frac{1}{\hbar} S \equiv \int_{t_i}^{t_f} dt \, \frac{m(\dot{\mathbf{x}})^2}{2} \tag{2}$$

3. DEFINITION OF THE WAVE FUNCTION. BORN'S PROBABILISTIC INTERPRETATION

The stochastic ensemble is described by the probability density $\rho(t, \mathbf{x}, \mathbf{v}, \phi)$.

The amount of information contained in ρ exceeds the amount of information available to the experimentalist who prepares a quantum state (a quantum ensemble).

It is assumed in quantum mechanics that all information about a particle available in a quantum experiment and sufficient to make predictions for these experiments is contained in the wave function $\psi(\mathbf{x})$.

We assume the wave function characterizes a certain class of stochastic ensembles. So many different stochastic ensembles described by different probability densities ρ belong to the same quantum ensemble, described by the wave function ψ. Distributions belonging to the same class are distinguished by properties hidden in a quantum experiment (or, in other words unobservable).

The partition to the classes is defined as follows:

Definition. The stochastic ensemble with probability density $\rho(t, \mathbf{x}, \mathbf{v}, \phi)$ belongs to the class specified by the wave function $\psi(t, \mathbf{x})$ if

$$\psi(t, \mathbf{x}) = \int d\mathbf{v} \int d\phi \, e^{i\phi} \rho(t, \mathbf{x}, \mathbf{v}, \phi) = \int d\phi \, e^{i\phi} \rho(t, \mathbf{x}, \phi) \tag{3}$$

where

$$\rho(t, \mathbf{x}, \phi) \equiv \int d\mathbf{v} \, \rho(t, \mathbf{x}, \mathbf{v}, \phi)$$

From this definition we can deduce Born's expression for the probability to

[2] The derivative $\dot{X}(t)$ of a generally nondifferentiable path $X(t)$ appeared in (2) in the same sense as in the ordinary path integral in Feynman and Hibbs (1965).

find a particle at the place $\mathbf{x}$ is $p(\mathbf{x}) = |\psi(\mathbf{x})|^2$

Theorem.

$$p(\mathbf{x}) = \langle \rho(\mathbf{x}) \rangle_\psi = a|\psi(\mathbf{x})|^2 \tag{4}$$

where a is constant and $\langle\ \rangle_\psi$ is average on the class defined by ψ and

$$\rho(\mathbf{x}) \equiv \int d\phi\, \rho(\mathbf{x}, \phi) \tag{5}$$

The proof, based on the central limit theorem is given in Appendix.

A probability of the other observables (for example momentum) can be obtained as in Ghirardi et al. (1978), Section 4.

4. DERIVATION OF SCHRÖDINGER EQUATION FOR A FREE PARTICLE

From the definition of wave function (3) and from the transition probability kernel $p(t_i, \mathbf{x}_i, \mathbf{v}_i, \phi_i | t_f, \mathbf{x}_f, \mathbf{v}_f, \phi_f)$ we can obtain the propagator for $\psi(x)$.

Let us indicate with subscripts i and f the quantities at the moments t_i and t_f ($t_i < t_f$). Then we have

$$\psi(t_f, \mathbf{x}_f) = \int d\mathbf{v}_f \int d\phi_f\, e^{i\phi_f} \rho(t_f, \mathbf{x}_f, \mathbf{v}_f, \phi_f) \tag{6}$$

$$\psi(t_i, \mathbf{x}_i) = \int d\mathbf{v}_i \int d\phi_i\, e^{i\phi_i} \rho(t_i, \mathbf{x}_i, \mathbf{v}_i, \phi_i) \tag{7}$$

From the definition of the transition probability kernel we have

$$\begin{aligned} \rho(t_f, \mathbf{x}_f, \mathbf{v}_f, \phi_f) &= \int d\mathbf{x}_i \int d\mathbf{v}_i \int d\phi_i\, p(t_i, \mathbf{x}_i, \mathbf{v}_i, \phi_i | t_f, \mathbf{x}_f, \mathbf{v}_f, \phi_f)\, \rho(t_i, \mathbf{x}_i, \mathbf{v}_i, \phi_i) \\ &= \int d\mathbf{x}_i \int d\mathbf{v}_i \int d\phi_i \int_{\mathbf{x}_i(t_i)}^{\mathbf{x}_f(t_f)} \mathscr{D}(\mathbf{x}(t))\, \delta(\mathbf{v}_f - \mathbf{v}_i - (\dot{\mathbf{x}}_f - \dot{\mathbf{x}}_i)) \\ &\quad \times \delta\left(\phi_f - \phi_i - \frac{1}{\hbar} S\right) \rho(t_i, \mathbf{x}_i, \mathbf{v}_i, \phi_i) \end{aligned} \tag{8}$$

Substituting (7) in (5) we obtain

$$\psi(t_f,\mathbf{x}_f)=\int d\mathbf{v}_f\int d\phi_f e^{i\phi_f}\int d\mathbf{x}_i\int d\mathbf{v}_i\int d\phi_i\int_{x_i(t_i)}^{x_f(t_f)}\mathscr{D}(\mathbf{x}(t))\delta\left(\phi_f-\phi_i-\frac{1}{\hbar}S\right)$$

$$\times\,\delta\big(\mathbf{v}_f-\mathbf{v}_i-(\dot{\mathbf{x}}_f-\mathbf{x}_i)\big)\rho(t_i,\mathbf{x}_i,\mathbf{v}_i,\phi_i) \tag{9}$$

Now let us use $\delta(\phi_f-\phi_i-(1/\hbar)S)$ and $\delta(\mathbf{v}_f-\mathbf{v}_i-(\dot{\mathbf{x}}_f-\dot{\mathbf{x}}_i))$ to perform the integrations on $\mathbf{v}_f$ and ϕ_f

$$\psi(t_f,\mathbf{x}_f)=\int d\mathbf{x}_i\int d\mathbf{v}_i\int d\phi_i\int_{\mathbf{x}_i(t_i)}^{\mathbf{x}_f(t_f)}\mathscr{D}(\mathbf{x}(t))e^{i[\phi_i+(1/\hbar)S]}\rho(t_i,\mathbf{x}_i,\mathbf{v}_i,\phi_i)$$

$$=\int d\mathbf{x}_i\left[\int_{\mathbf{x}_i(t_i)}^{\mathbf{x}_f(t_f)}\mathscr{D}(\mathbf{x}(t))e^{i(1/\hbar)S}\right]\int dv_i\int d\phi_i\, e^{i\phi_i}\rho(t_i,\mathbf{x}_i,\mathbf{v}_i,\phi_i)$$

$$=\int d\mathbf{x}_i\, P(t_i,\mathbf{x}_i|t_f,\mathbf{x}_f)\psi(t_i,\mathbf{x}_i) \tag{10}$$

where

$$P(t_i,\mathbf{x}_i|t_f,\mathbf{x}_f)=\int_{\mathbf{x}_i(t_i)}^{\mathbf{x}_f(t_f)}\mathscr{D}(\mathbf{x}(t))e^{i(1/\hbar)S}$$

is exactly the Feynman integral for the propagator of the Schrödinger equation for a free particle. Therefore $\psi(t,\mathbf{x})$ obeys the Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}\psi(t,\mathbf{x})=-\frac{\hbar^2}{2m}\Delta\psi(t,\mathbf{x}) \tag{11}$$

5. A PARTICLE IN THE ELECTROMAGNETIC FIELD. GAUGE INVARIANCE

In order to describe a particle in a nonquantized electromagnetic field we modify Assumption 2:

Assumption 2′. If a particle moves along the path $x(t)$ $t_i<t<t_f$ the change of the phase is

$$\phi_f-\phi_i=\frac{1}{\hbar}S_{\text{free}}+e\int_{t_i}^{t_f}dt\,[\Phi(\mathbf{x}(t))+\dot{\mathbf{x}}(t)\mathbf{A}(\mathbf{x}(t))] \tag{12}$$

So the presence of an electromagnetic field leads to the additional phase velocity $e\Phi(x)$ independent of particle velocity and to the additional phase velocity $e\dot{\mathbf{x}}\mathbf{A}(x)$ depending on a particle velocity:

$$\frac{d\phi}{dt} = e[\Phi(\mathbf{x}) + \dot{\mathbf{x}}\mathbf{A}(\mathbf{x})] \tag{13}$$

So far we assumed the phase ϕ is defined at any point $\mathbf{x}$ with respect to some arbitrary reference phase. Let us change the reference phase by the amount $\Lambda(\mathbf{x})$ (it can depend on the position $\mathbf{x}$). In other words we make the change

$$\phi \to \tilde{\phi} = \phi + \Lambda(\mathbf{x}) \tag{14}$$

Then

$$\rho(t,\mathbf{x},\phi) \to \tilde{\rho}(t,\mathbf{x},\phi) = \rho\big(t,\mathbf{x},\tilde{\phi} - \Lambda(\mathbf{x})\big) \tag{15}$$

and

$$\psi(t,\mathbf{x}) \to \tilde{\psi}(t,\mathbf{x}) \equiv \int d\tilde{\phi}\, e^{i\tilde{\phi}}\tilde{\rho}(t,\mathbf{x},\tilde{\phi}) = \int d\tilde{\phi}\, e^{i\tilde{\phi}}\rho\big(t,\mathbf{x},\tilde{\phi} - \Lambda(\mathbf{x})\big) \tag{16}$$

and the change of the integration variable $\phi \to \tilde{\phi} = \phi - \Lambda(x)$ leads to

$$\tilde{\psi}(t,\mathbf{x}) = \int d\phi\, e^{i[\phi - \Lambda(x)]}\rho(t,\mathbf{x},\phi) = e^{-i\Lambda(\mathbf{x})}\int d\phi\, e^{i\phi}\rho(t,\mathbf{x},\phi) = e^{-i\Lambda(\mathbf{x})}\psi(t,\mathbf{x}) \tag{17}$$

From (13) we have

$$\frac{d\tilde{\phi}}{dt} = \frac{d\phi}{dt} + \frac{\partial\Lambda}{\partial t} + \dot{\mathbf{x}}\,\nabla\Lambda = e[\Phi + \dot{\mathbf{x}}\mathbf{A}] \tag{18}$$

This implies

$$\tilde{\Phi} = \Phi - \frac{1}{e}\frac{\partial\Lambda}{\partial t}$$

$$\tilde{\mathbf{A}} = \mathbf{A} - \frac{1}{e}\nabla\Lambda \tag{19}$$

(17) and (19) are $U(1)$ gauge transformations for the particle wave function and electromagnetic potential.

6. CONCLUDING REMARKS. RELATIVISTIC EXTENSIONS

Let us reformulate Assumption 2 for the relativistic case.

Assumption 2 (relativistic case). If a particle moves along the path $x(t)$ the change of the phase is proportional to proper time:

$$\phi_f - \phi_i = -\frac{mc^2}{\hbar}(\tau_f - \tau_i) = \frac{mc^2}{\hbar}\int_{t_i}^{t_f}\left[1 - \frac{(\dot{\mathbf{x}})^2}{c^2}\right]^{1/2} dt \tag{20}$$

The constant $mc^2/\hbar \sim 10^{-21} 1/\text{sec}$ is a (constant) phase velocity of the free motion. The nonrelativistic limit of (20) is

$$\phi_f - \phi_i = \frac{mc^2}{\hbar}\int_{t_i}^{t_f} dt \left[1 - \frac{(\dot{\mathbf{x}})^2}{c^2}\right]^{1/2} \approx \frac{mc^2}{\hbar}\int_{t_i}^{t_f}\left[1 - \frac{(\dot{\mathbf{x}})^2}{2c^2}\right] dt$$

$$= \frac{mc^2}{\hbar}(t_f - t_i) - \frac{1}{\hbar}\int_{t_i}^{t_f} dt\, \frac{(\dot{\mathbf{x}})^2}{2m} = \frac{mc^2}{\hbar}(t_f - t_i) - \frac{1}{\hbar}S \tag{21}$$

The rest mass term $(mc^2/h)(t_f - t_i)$ is absorbed in the definition of the nonrelativistic wave function in the usual transition from Dirac to Schrödinger equation (see, for example, Bjorken and Drell, 1964). We conjecture that for the antiparticle we have the change of phase in the opposite direction

$$\phi_f - \phi_i = -\frac{mc^2}{\hbar}(\tau_f - \tau_i) \tag{22}$$

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APPENDIX: DERIVATION OF BORN'S PROBABILISTIC INTERPRETATION OF THE WAVE FUNCTION THEOREM

The probability density $p(\mathbf{x})$ to find particle in $\mathbf{x}$ in quantum state $\psi(\mathbf{x})$ is

$$p(\mathbf{x}) = \langle \rho(\mathbf{x}) \rangle_\psi = a|\psi(\mathbf{x})|^2 \tag{A.1}$$

where a is a constant, $\langle\ \rangle_\psi$ is an average on the class ψ, and $\rho(\mathbf{x})$ is defined by (5)

Proof. To "discretize" the problem we divide the coordinate space into many regions A_i of a sufficiently small volume v that the wave function $\psi(\mathbf{x}) = R(\mathbf{x})e^{i\chi(\mathbf{x})}$ is essentially constant in any region

$$\begin{aligned} R_i &\leqslant R(\mathbf{x}) \leqslant R_i + \Delta R \\ \chi_i &\leqslant \chi(\mathbf{x}) \leqslant \chi_i + \Delta\chi \end{aligned} \tag{A.2}$$

or in short, $\psi(\mathbf{x}) \approx R_i e^{i\chi_i} \equiv \psi_i$, where ΔR and $\Delta\chi$ are small. The phase ϕ will take values $\phi = 2\pi\lambda/\Gamma$, $\lambda = 0,\dots\Gamma-1$; $\Gamma \gg 1$. Therefore in this "discretized" version a stochastic ensemble is described by

$$p_{i\lambda} \equiv \int_{2\pi(\lambda-1)/\Gamma}^{2\pi\lambda/\Gamma} d\phi \int_{A_i} \rho(\mathbf{x},\phi)\, d\mathbf{x} \tag{A.3}$$

and a quantum state by ψ_i. To give precise meaning to the expectation value $\langle\ \rangle_\psi$ in (A.1) we have to define a measure on the class of stochastic ensembles ψ_i defined by the wave function $\psi(\mathbf{x})$. As in any case of the definition of measure on a function space we must discretize the values that these functions take. In this case, we assume that the probability $p_{i\lambda}$ takes discrete values $n_i = N_{i\lambda}/N$; $N_{i\lambda} = 0,1,\dots,N$; $N \gg 1$. Therefore, the probability $\rho_{i\lambda}$ of finding a particle at the place i and phase λ is some number $N_{i\lambda}$ of "quanta" of probability $1/N$. The measure on the class ψ_i of stochastic ensembles $\rho_{i\lambda}$ is defined by the following assumption: the quanta of probability are distributed randomly between different regions A_i and phases λ, but with the constraint

$$\psi_i \approx \sum_\lambda \frac{N_{i\lambda}}{N} e^{i2\pi\lambda/\Gamma} \tag{A.4}$$

where $\approx$ is defined in (A.2).

The constraints (A.4) in different regions A_i are independent. Let us find the probability $p(N_i)$ that there is $N_i = \Sigma_\lambda N_{j\to\lambda}$ quanta of probability in the region A_i irrespective of number of quanta in other regions.

For this purpose, we shall use the isomorphism between vectors in $\mathbb{R}^2$ and complex numbers $\mathbb{C}$

$$\begin{gathered} \psi_i \equiv R_i e^{i\chi_i} \in \mathbb{C} \to \psi_i \equiv R_i(\cos\chi_i, \sin\chi_i) \in \mathbb{R}^2 \\ \frac{1}{N} e^{i2\pi\lambda/\Gamma} \in \mathbb{C} \to \lambda \equiv \frac{1}{N}\left(\cos\left(\frac{2\pi\lambda}{\Gamma}\right), \sin\left(\frac{2\pi\lambda}{\Gamma}\right)\right) \in \mathbb{R}^2 \end{gathered} \tag{A.5}$$

So (A.4) takes form

$$\psi_i \approx \sum_{\lambda} N_{i\lambda}\lambda = \sum_{j=1}^{N_i} \lambda_j \tag{A.4$'$}$$

where λ_j is the "direction of the jth quantum." In this form the problem is equivalent to the problem of finding the probability to reach the place ψ_i by N_i steps in two-dimensional random walk with constant step length $-1/N$ and random direction (we take here $\Gamma \to \infty$).

The probability $p(N_i)$ for large N_i is obtained from the central limit theorem (Bjorken and Drell, 1964):

$$P(N_i)\,\Delta R\,\Delta\chi = \text{const}\frac{N}{N_i}e^{-N|\psi_i|^2/N_i}\,\Delta R\,\Delta\chi \tag{A.6}$$

Consequently, the probability for N_i quanta to be in A_i for all i is

$$p\left\{N_i|\sum_i N_i = N\right\} = \prod_i p(N_i) \tag{A.7}$$

We have to find the mean value of the number of quanta in some particular region A_1

$$\langle N_1\rangle_\psi = \frac{\displaystyle\sum_{\{N_i|\Sigma N_i = N\}} N_1\, p\left\{N_i|\sum_i N_i = N\right\}}{\displaystyle\sum_{\{N_i|\Sigma N_i = N\}} p\left\{N_i|\sum_i N_i = N\right\}} \tag{A.8}$$

where $\Sigma_{\{N_i|\Sigma_i N_i = N\}}$ is the sum on all possible distributions of quanta among the regions A_i. The way to calculate (A.8) from (A.6) is analogous to the method used in statistical physics to calculate a mean value of a microsystem contained in a thermostat.[3]

If we denote by $p(M)$ the probability that M quanta are in all the regions *except* A_1, then $p(M)$ is the probability that the sum of a very large number of independent random variables defined by (A.6) is M. It has a very sharp peak at

$$\overline{M} = \sum_{i\neq 1} \overline{N_i} \tag{A.9}$$

[3] N is analogous to the total energy of the microsystem and of the thermostat and N_i are analogous to the energies of the microsystems. See for example, Landau and Lifshitz (1980).

where mean value $\bar{N}_i$ can be calculated from (A.6)

$$\bar{N}_i \equiv \frac{\int \frac{N}{N_i} N_i e^{-N|\psi_i|^2/N_i} dN_i}{\int \frac{N}{N_i} e^{-N|\psi_i|^2/N_i} dN_i} = N|\psi_i|^2 \tag{A.10}$$

Therefore

$$\langle N_1 \rangle_\psi = \int dM P(M) P(N_1 = N - M) \cdot N_1 \approx N - \sum_{i \neq 1} \bar{N}_i = \bar{N}_1 = N|\psi_1|^2 \tag{A.11}$$

Consequently, $p(\mathbf{x}) = (1/v)|\psi(\mathbf{x})|^2$ and ψ can be normalized to $\int d\mathbf{x}\, |\psi|^2 = 1$.

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